Geometric and ILP-based heuristics for the quadratic travelling salesman problem

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The well-known *travelling salesman problem* (*TSP*) asks for a shortest tour through all vertices of a graph with respect to the costs of the edges. The *quadratic travelling salesman problem* (*QTSP*) associates a cost value with every two edges traversed in succession. We consider a symmetric special case that arises in robotics, in which the quadratic costs correspond to the *turning angles* or—more generally—to a linear combination of the *turning angles* and *Euclidean distances*. We introduce two heuristic approaches, exploiting the geometric properties of a good tour and the impressive performance of current ILP-solvers.

If the quadratic costs correspond to the turning angles, optimal tours usually have the form of large circles or spiral shapes. In each step of the first algorithm, a *convex hull* is built and its corresponding vertices are removed from the graph. This step is iteratively repeated until a pre-specified small number of vertices remains untouched in the centre. After computing an optimal tour on these central vertices, we obtain a set of nested subtours, which are subsequently patched into one single tour.

This idea is improved by introducing the so-called *lens neighbourhood*. Generally, we position lenses, i.e. quasi-elliptic areas, between the two end-vertices of an edge in such a way that their two positive curvatures intersect at these vertices. The motivation for this new idea comes from the fact that the inclusion of any vertex inside such a lens causes an additional turning angle that is upper-bounded by the gradient of the curvatures tangents at the edges end-vertices. Thus we iteratively enlarge each convex hull built by considering the lens neighbourhood for all its edges. This procedure leads to a smaller number of subtours, which contain a larger number of vertices.

Our second algorithm uses an *LP relaxation* and a *rounding procedure* to obtain a set of paths and isolated vertices. Afterwards, these paths are optimally patched into a

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single tour by means of a small auxiliary TSP instance, which is solved by an ILP-solver. Finally, the resulting tour is enlarged by adding the remaining isolated vertices using a cheapest insertion heuristic. Obviously, the number of isolated vertices should be as small as possible since the paths are patched in an optimal way, but the isolated vertices are included heuristically. Therefore we present some further algorithmical enhancements in order to decrease the number of such vertices.

All constructive results can be improved by running a classical *3-opt improvement heuristic*. We provide exhaustive computational tests, which illustrate that both introduced algorithms significantly outperform the best-known heuristic approaches from the literature with a dominance of the LP based approach.